

## ON CLOSED SETS WITH CONVEX PROJECTIONS UNDER NARROW SETS OF DIRECTIONS

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**ABSTRACT.** Dijkstra, Goodsell, and Wright have shown that if a nonconvex compactum in  $\mathbb{R}^n$  has the property that its projection onto *all*  $k$ -dimensional planes is convex, then the compactum contains a topological copy of the  $(k-1)$ -sphere. This theorem was extended over the class of unbounded closed sets by Barov, Cobb, and Dijkstra. We show that the results in these two papers remain valid under the much weaker assumption that the collection of projection directions has a nonempty interior.

### 1. INTRODUCTION

Consider the vector space  $\mathbb{R}^n$  for  $n \geq 3$ . Let us call the image of a set  $X \subset \mathbb{R}^n$  under an orthogonal projection onto a hyperplane a *shadow* of  $X$ . Borsuk [3] has shown that there exist Cantor sets in  $\mathbb{R}^n$  such that all their shadows contain  $(n-1)$ -dimensional convex bodies. In contrast, Cobb [4] showed that every compactum  $C$  in  $\mathbb{R}^n$  with the property that all its shadows are convex bodies contains an arc. Dijkstra, Goodsell, and Wright [5] improved on this result by showing that such a  $C$  must contain an  $(n-2)$ -sphere, so in this case projections cannot raise dimension by more than one. Barov, Cobb, and Dijkstra [1] were subsequently able to construct an extension of the result over the class of unbounded closed sets. Note that in these papers we are dealing with shadows in *all* directions. Remarkably, in this paper we show that the results in [5] and [1] remain valid if we make the much weaker assumption that the collection of projection directions that produce convex shadows has a nonempty interior. Thus we see that it suffices to have a ‘narrow beam’ of directions that produce convex shadows to find  $(n-2)$ -manifolds in the sets  $C$ .

We now formulate one of the incarnations of our main result. If  $A \subset \mathbb{R}^n$ , then  $\langle A \rangle$  is the convex hull and  $\overline{A}$  is the closure. If  $L$  is a linear subspace of  $\mathbb{R}^n$ , then  $L^\perp$  is the orthocomplement of  $L$  and  $\psi_L$  is the orthogonal projection of  $\mathbb{R}^n$  along  $L$  onto  $L^\perp$ . The space  $\mathbb{L}_k^n$  consists of all  $k$ -dimensional linear subspaces of  $\mathbb{R}^n$  with the natural topology; see Definition 2. A  $k$ -plane in  $\mathbb{R}^n$  is a  $k$ -dimensional affine subspace.  $S^{i-1}$  is the unit sphere in  $\mathbb{R}^i$ .

**Theorem 1.** *Let  $0 < k < n$ , let  $C$  be a closed nonconvex subset of  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be open in  $\mathbb{L}_{n-k}^n$ . Let  $\psi_{P^*}(\langle C \rangle) \neq (P^*)^\perp$  for some  $P^* \in \mathcal{P}$  and let  $\overline{\psi_P(C)}$  be convex*

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for every  $P \in \mathcal{P}$ . If  $\langle C \rangle$  contains no  $k$ -plane, then  $C$  contains a closed set that is homeomorphic to either

- (i)  $\mathbb{R}^{k-1}$  or
- (ii)  $S^i \times \mathbb{R}^{k-i-1}$  for some  $i \in \{1, 2, \dots, k-1\}$ .

The method used in [5] and [1] consists of finding a high-dimensional derived face of  $\overline{\langle C \rangle}$  of which it can be proved that its boundary is in  $C$ . This method can also be applied in the situation that  $\mathcal{P}$  is a proper subset of  $\mathbb{L}_{n-k}^n$  (see Theorem 16) but only if the face is consistent with  $\mathcal{P}$ , by which we mean that it is contained in a supporting hyperplane  $H$  such that  $H + P = H$  for some  $P \in \mathcal{P}$ . If such faces do not exist, then the method in [5] and [1] breaks down. We solve this problem by proving that if there are no high-dimensional faces of  $\overline{\langle C \rangle}$  that are consistent with  $\mathcal{P}$ , then there exists a high-dimensional projection of  $\overline{\langle C \rangle}$  such that some open subset of its boundary can be ‘lifted’ back up to  $C$ ; see the proof of Theorem 17. The procedure for finding this open set takes up most of §3 and §4.

Note that Theorem 1 deals with the retrieval of information about a geometric object from data about its projections which places the result in the field of Geometric Tomography; see Gardner [7] for background information.

The paper is arranged as follows. In §2 we define the main concepts and establish some basic properties. §3 contains a collection of lemmas that prepare the ground for §4, where we prove our main theorems. We finish in §5 with a discussion of examples that show that our main results are sharp.

## 2. DEFINITIONS AND PRELIMINARIES

In  $\mathbb{R}^n$  we shall use the standard dot product:  $u \cdot v = \sum_{i=1}^n u_i v_i$  for  $u = (u_1, \dots, u_n)$  and  $v = (v_1, \dots, v_n)$  elements of  $\mathbb{R}^n$ . For  $m \geq 0$  the standard  $m$ -sphere is  $S^m = \{u \in \mathbb{R}^{m+1} : \|u\| = 1\}$ . An  $m$ -sphere is any space that is homeomorphic to  $S^m$ .

Let  $V$  be a finite-dimensional vector space with inner product  $x \cdot y$  and zero vector  $\mathbf{0}$ . Let  $n = \dim V$ . The norm on  $V$  is given by  $\|u\| = \sqrt{u \cdot u}$ , and the metric  $d$  is given by  $d(u, v) = \|v - u\|$ . Let  $A$  be a subset of  $V$ . We have that  $[A]$  denotes the linear hull,  $\text{aff } A$  the affine hull,  $\langle A \rangle$  the convex hull,  $\overline{A}$  the closure, and  $\text{int } A$  the interior of  $A$  in  $V$ . A closed and convex set  $A$  with  $\text{int } A \neq \emptyset$  is called a *convex body* in  $V$ . Also,  $\partial A$  means the relative boundary of  $A$ , that is, the boundary with respect to  $\text{aff } A$ , and we define  $A^\circ = A \setminus \partial A$ . Note that if  $A$  is convex in a finite-dimensional space, then  $A^\circ \neq \emptyset$  and  $\overline{A^\circ} \subset A$ ; thus  $\overline{A^\circ}$  is a convex body in  $\text{aff } A$ . We also define the linear space

$$A^\perp = \{x \in V : x \cdot y = 0 \text{ for all } y \in A\}.$$

If  $A$  is a linear space, then  $A^{\perp\perp} = A$ , and  $A^\perp$  is called the *orthocomplement* of  $A$ .

A  $k$ -space in  $V$  is a  $k$ -dimensional linear subspace of  $V$ . A  $k$ -plane in  $V$  is a  $k$ -dimensional affine subspace of  $V$ . Let  $H$  be a *hyperplane* in  $V$ , that is, an  $(n-1)$ -plane. The two components of  $V \setminus H$  are called the *sides* of  $H$ . We say that  $H$  *cuts* a subset  $A$  of  $V$  if  $A$  contains points on both sides of  $H$ . We say that a hyperplane  $H$  in  $V$  is *supporting to*  $A$  at  $x$  if  $x \in H$  and  $H$  does not cut  $A$ . A subset  $L$  of  $V$  is called a *halfspace* of  $V$  if it is the union of a hyperplane and one of its sides. A  $k$ -halfplane is a halfspace of a  $k$ -plane. For each  $a \in V \setminus \{\mathbf{0}\}$  we define the  $(n-1)$ -space

$$H_a = \{x \in V : a \cdot x = 0\}.$$

**Definition 1.** If  $L$  is an affine subspace in  $\mathbb{R}^n$ , then  $\psi_L : \mathbb{R}^n \rightarrow L^\perp$  denotes the orthogonal projection along  $L$  onto  $L^\perp$  defined by the conditions  $\psi_L(x) - x \in L^{\perp\perp}$  and  $\psi_L(x) \in L^\perp$  for each  $x \in \mathbb{R}^n$ .

**Definition 2.**  $\mathcal{K}(V)$  stands for all nonempty compact subsets of  $V$ . Recall that the Hausdorff metric  $d_H$  on  $\mathcal{K}(V)$  associated with  $d$  is defined as follows:

$$d_H(A, B) = \sup\{d(x, A), d(y, B) : x \in B \text{ and } y \in A\}.$$

We let  $\mathbb{L}_m(V)$  stand for the collection of all  $m$ -dimensional linear subspaces of  $V$ . Consider the ball  $\mathbb{B} = \{v \in V : \|v\| \leq 1\}$ . We topologize  $\mathbb{L}_m(V)$  by defining a metric  $\rho$  on  $\mathbb{L}_m(V)$ :

$$\rho(L_1, L_2) = d_H(L_1 \cap \mathbb{B}, L_2 \cap \mathbb{B}).$$

We let  $\mathbb{L}_m^n$  stand for  $\mathbb{L}_m(\mathbb{R}^n)$ . Note that  $\mathbb{L}_1^n$  is the projective space of dimension  $n-1$ . The singletons  $\mathbb{L}_0^n = \{\{\mathbf{0}\}\}$  and  $\mathbb{L}_n^n = \{\mathbb{R}^n\}$  are of course not very interesting, but it is sometimes useful to have them available.

**Lemma 2.** Let  $0 < m < n$ ,  $\varepsilon > 0$ ,  $L \in \mathbb{L}_m^n$ , and let  $v_1, \dots, v_m$  be a basis for  $L$ . Then there is a  $\delta > 0$  such that for every set  $F = \{v'_1, \dots, v'_m\} \subset \mathbb{R}^n$  with  $\|v'_i - v_i\| < \delta$  for every  $i$  we have  $\rho([F], L) < \varepsilon$ .

*Proof.* We may assume that  $\varepsilon < 1$ . Since  $v_1, \dots, v_m$  are linearly independent we have that

$$A = \left\{ (a_1, \dots, a_m) \in \mathbb{R}^m : \left\| \sum_{i=1}^m a_i v_i \right\| \leq 2 \right\}$$

is compact. Note that there exists a  $\delta > 0$  such that if  $\|v'_i - v_i\| < \delta$  for every  $i$ , then  $\|\sum_{i=1}^m a_i(v'_i - v_i)\| < \varepsilon/2$  for each  $(a_1, \dots, a_m) \in A$ . Let  $F = \{v'_1, \dots, v'_m\} \subset \mathbb{R}^n$  be such that  $\|v'_i - v_i\| < \delta$  for every  $i$  and let  $a = \sum_{i=1}^m a_i v_i$  be an element of  $L \cap \mathbb{B}$ . Since  $\|a\| \leq 1$  we have  $(a_1, \dots, a_m) \in A$ . Put  $a' = \sum_{i=1}^m a_i v'_i$  and note that  $\|a'\| \leq \|a\| + \|a' - a\| < 1 + \varepsilon/2$ . Let  $b = a' / \max\{1, \|a'\|\}$  and note that  $b \in [F] \cap \mathbb{B}$  and

$$\begin{aligned} \|b - a\| &\leq \|b - a'\| + \|a' - a\| = \|a'\| (1 - (\max\{1, \|a'\|\})^{-1}) + \|a' - a\| \\ &< (1 + \varepsilon/2)(1 - (1 + \varepsilon/2)^{-1}) + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus  $d(a, [F] \cap \mathbb{B}) < \varepsilon$ .

Now let  $a' = \sum_{i=1}^m a_i v'_i$  be an element of  $[F] \cap \mathbb{B}$ . Put  $a = \sum_{i=1}^m a_i v_i \in L$ . We prove that  $\|a\| \leq 2$ . We may assume that  $\|a\| \neq 0$ , and we consider  $a/\|a\|$ . Since  $(a_1, \dots, a_m)/\|a\|$  is obviously an element of  $A$  we have that  $\|a' - a\|/\|a\| < \varepsilon/2 < 1/2$ . Thus  $\|a\| - \|a'\| \leq \|a' - a\| \leq \|a\|/2$ , which means that  $\|a\| \leq 2\|a'\| \leq 2$ . Consequently, we have that  $(a_1, \dots, a_m) \in A$  and hence  $\|a' - a\| < \varepsilon/2$ . The fact that  $(a_1, \dots, a_m) \in A$  also means that  $F$  consists of  $m$  independent vectors, so  $[F] \in \mathbb{L}_m^n$ . Defining  $b = a/\max\{1, \|a\|\}$  the same argument as above gives that  $d(a', L \cap \mathbb{B}) \leq \|b - a'\| < \varepsilon$ . Thus we have  $\rho([F], L) < \varepsilon$ .  $\square$

The next lemma shows that the converse of Lemma 2 also holds. Thus Lemmas 2 and 3 give us an alternative way to define the topology on  $\mathbb{L}_m^n$ .

**Lemma 3.** Let  $0 < m < n$ ,  $\varepsilon > 0$ ,  $L \in \mathbb{L}_m^n$ , and let  $v_1, \dots, v_m$  be a basis for  $L$ . Then there is a  $\delta > 0$  such that for every  $P \in \mathbb{L}_m^n$ , with  $\rho(L, P) < \delta$ , there is a basis  $\{v'_1, \dots, v'_m\}$  for  $P$  such that  $\|v'_i - v_i\| < \varepsilon$  for every  $1 \leq i \leq m$ .

*Proof.* Let  $s = \max\{\|v_i\| : i = 1, \dots, m\}$ . Since determinants are continuous we can choose a  $\delta \in (0, \varepsilon/s)$  such that every set  $\{w_1, w_2, \dots, w_m\} \subset \mathbb{R}^n$ , with  $\|\frac{v_i}{\|v_i\|} - w_i\| < \delta$  for every  $1 \leq i \leq m$ , consists of  $m$  linearly independent vectors. Let  $P$  be such that  $\rho(L, P) < \delta$  and select for each  $i \in \{1, \dots, m\}$  a vector  $v_i^* \in P$  with  $\|\frac{v_i}{\|v_i\|} - v_i^*\| < \delta$ . Put  $v'_i = \|v_i\|v_i^*$ . Note that the  $v_i^*$ 's are independent and thus they and the  $v'_i$ 's both form bases for  $P$ . Let us show that the  $v'_i$ 's are as required. Indeed,

$$\|v_i - v'_i\| = \|v_i - \|v_i\|v_i^*\| = \|v_i\| \cdot \left\| \frac{v_i}{\|v_i\|} - v_i^* \right\| < \|v_i\|\delta < \varepsilon.$$

This completes the proof.  $\square$

**Definition 3.** Let  $0 \leq i \leq m < n$  and let  $\mathcal{P}$  be a subset of  $\mathbb{L}_m^n$ . If  $L \in \mathbb{L}_i^n$ , then we define

$$\mathcal{P}_L = \{N \in \mathbb{L}_{m-i}(L^\perp) : N + L \in \mathcal{P}\}.$$

**Corollary 4.** Let  $0 \leq i \leq m < n$  and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_m^n$ . If  $L \in \mathbb{L}_i^n$ , then  $\mathcal{P}_L$  is open in  $\mathbb{L}_{m-i}(L^\perp)$ .

*Proof.* If  $i = 0$ , then  $\mathcal{P}_L = \mathcal{P}$ , and if  $i = m$ , then  $\mathcal{P}_L = \emptyset$  or  $\mathcal{P}_L = \mathbb{L}_{m-i}(L^\perp)$ , so there is nothing to prove. Let  $0 < i < m$  and consider an  $N \in \mathcal{P}_L$ . Then there exist independent vectors  $v_1, v_2, \dots, v_m$  such that  $L = [\{v_1, \dots, v_i\}]$  and  $N = [\{v_{i+1}, \dots, v_m\}]$  and hence  $N + L = \psi_L^{-1}(N) = [\{v_1, v_2, \dots, v_m\}]$ . By Lemma 2 we can find an  $\varepsilon$  such that

$$[F] \in \mathcal{P} \text{ whenever } F = \{v'_1, v'_2, \dots, v'_m\} \text{ with } \|v_j - v'_j\| < \varepsilon.$$

Now, we can apply Lemma 3 and find a  $\delta$  such that if  $P \in \mathbb{L}_{m-i}(L^\perp)$  with  $\rho(P, N) < \delta$ , then there is a basis  $\{v'_{i+1}, \dots, v'_m\}$  for  $P$  such that  $\|v_j - v'_j\| < \varepsilon$  for  $i+1 \leq j \leq m$ . Consequently, the open neighbourhood

$$\{P \in \mathbb{L}_{m-i}(L^\perp) : \rho(P, N) < \delta\}$$

of  $N$  in  $\mathbb{L}_{m-i}(L^\perp)$  is a subset of  $\mathcal{P}_L$ . That completes the proof.  $\square$

**Definition 4.** Let  $B$  be a closed and convex set in  $V$ . A subset  $F$  of  $B$  is called a *face* of  $B$  if there is a hyperplane  $H$  of aff  $B$  that does not cut  $B$  with the property  $F = B \cap H$ . Note that  $F$  is also closed and convex and that  $\dim F < \dim B$ . If  $F$  is a face of  $B$  we write  $F \prec B$ . We say that a subset  $F$  of  $B$  is a *derived face* of  $B$  if  $F = B$  or there exists a sequence  $F = F_1 \prec F_2 \prec \dots \prec F_m = B$  for some  $m \in \mathbb{N}$ .

*Remark 1.* Let  $F \prec B$  and put  $m = \dim F$ ,  $H_m = \text{aff } F$ ,  $k = \dim B$ , and  $H_k = \text{aff } B$ . There is a hyperplane  $H_{k-1}$  of  $H_k$  that does not cut  $B$  and with the property  $F = B \cap H_{k-1}$ . If  $H_{k-1} \neq \text{aff } F$ , then  $m < k-1$ , and we can fill in the missing dimensions and construct a sequence  $H_m \subset H_{m+1} \subset \dots \subset H_k$  of affine spaces such that  $\dim H_i = i$  for  $i \in \{m, \dots, k\}$ . Note that  $H_{i-1}$  is a hyperplane in  $H_i$  that does not cut  $B \cap H_i$  for  $i \in \{m+1, \dots, k\}$ .

*Remark 2.* We list a few facts concerning closed convex sets and hyperplanes; see [8, §2.2]. Let  $B$  be a closed and convex set in  $\mathbb{R}^n$ . If the interior of  $B$  is nonempty, then a hyperplane  $H$  cuts  $B$  if and only if  $H$  meets the interior of  $B$ . Every point in  $\partial B$  is contained in a hyperplane  $H$  of aff  $B$  that does not cut  $B$ . In other words,  $\partial B$  equals the union of the faces of  $B$ .

**Definition 5.** Let  $\mathcal{P}$  be a collection of linear subspaces of a vector space  $V$ . We say that an affine subspace  $H$  of  $V$  is *consistent with  $\mathcal{P}$*  if there is a  $P \in \mathcal{P}$  such that  $z + P \subset H$  for  $z \in H$ .

**Definition 6.** Let  $B$  be a convex and closed subset of  $V$ , and let  $\mathcal{P}$  be a collection of linear subspaces of  $V$ . A subset  $F$  of  $B$  is called a  $\mathcal{P}$ -face of  $B$  if  $F = B \cap H$  for some hyperplane  $H$  of  $\mathbb{R}^n$  that does not cut  $B$  and that is consistent with  $\mathcal{P}$ . A *derived  $\mathcal{P}$ -face* is a derived face of a  $\mathcal{P}$ -face. If  $0 < k < \dim V$ , then we define the set  $\mathcal{E}_k(B, \mathcal{P})$  as the closure of

$$\bigcup \{F : F \text{ is a derived } \mathcal{P}\text{-face of } B \text{ with } \dim F < k\}.$$

**Definition 7.** Let  $A_1$  and  $A_2$  be subsets of  $V$  and let  $\mathcal{P}$  be a collection of linear subspaces of  $V$ .  $A_1$  is called a  $\mathcal{P}$ -imitation of  $A_2$  if  $\overline{\psi_P(A_1)} = \overline{\psi_P(A_2)}$  for each  $P \in \mathcal{P}$ .  $A_1$  is called a *weak  $\mathcal{P}$ -imitation* of  $A_2$  if  $\overline{\psi_P(A_1)} = \overline{\psi_P(A_2)}$  for each  $P \in \mathcal{P}$ . If  $B$  is a closed convex subset of  $V$  and  $\mathcal{P}$  is a nonempty subset of  $\mathbb{L}_m(V)$ , then a point  $x \in B$  is called  $\mathcal{P}$ -extremal if  $x \in \mathcal{E}_{\dim V - m}(B, \mathcal{P})$ . We show in §5 that the  $\mathcal{P}$ -extremal points of  $B$  are precisely the points that the closed  $\mathcal{P}$ -imitations of  $B$  have in common.

**Definition 8.** A subset  $A$  of  $S^{n-1}$  is called *convex* if  $w \in A$  whenever  $w = \alpha u + \beta v \in S^{n-1}$  with  $\alpha, \beta \geq 0$  and  $u, v \in A$ .

**Definition 9.** Let  $X$  and  $Y$  be topological spaces and let  $2^Y$  stand for the collection of nonempty subsets of  $Y$ . A set-valued  $\varphi: X \rightarrow 2^Y$  is called *USC* (*upper semi-continuous*) if  $\varphi^{-1}(U) = \{x \in X : \varphi(x) \subset U\}$  is open in  $X$  for every open  $U$  in  $Y$ .

**Definition 10.** Let  $B$  be a convex and closed subset of  $\mathbb{R}^n$ , and we define a set-valued function  $\Phi: \mathbb{R}^n \setminus \text{int } B \rightarrow 2^{S^{n-1}}$  as follows:

$$\Phi(x) = \{a \in S^{n-1} : a \cdot (y - x) \leq 0 \text{ for every } y \in B\}.$$

In other words,  $\Phi(x)$  consists of all unit vectors  $a$  such that  $x + H_a$  is supporting to  $B$  and  $a$  points towards a side of  $x + H_a$  that does not contain points of  $B$ .

**Lemma 5.** *Let  $B$  be a closed and convex subset in  $\mathbb{R}^n$ . Then each  $\Phi(x)$  is nonempty, closed, and convex in  $S^{n-1}$ , and  $\Phi$  is a USC set-valued map. If  $B$  is a convex body, then no  $\Phi(x)$  contains antipodal vectors.*

*Proof.* It is clear that  $\Phi(x) \neq \emptyset$  because of the Hahn-Banach theorem. By the continuity of the dot product  $\Phi(x)$  is closed. Convexity is equally trivial.

We now prove that  $\Phi$  is USC. Let  $U$  be open in  $S^{n-1}$  and let  $x \in \mathbb{R}^n \setminus \text{int } B$  be such that there is a sequence  $x_1, x_2, \dots$  in  $\mathbb{R}^n \setminus (\text{int } B \cup \Phi^{-1}(U))$  that converges to  $x$ . Select for each  $i \in \mathbb{N}$  a  $u_i \in \Phi(x_i) \setminus U$ . Since  $S^{n-1}$  is compact we may assume that  $u_1, u_2, \dots$  converges to some  $u \in S^{n-1} \setminus U$ . Let  $y \in B$  and note that

$$u \cdot (y - x) = \lim_{i \rightarrow \infty} u_i \cdot (y - x_i) \leq 0.$$

Thus we have that  $u \in \Phi(x) \setminus U$  and hence  $x \notin \Phi^{-1}(U)$ . Thus  $\mathbb{R}^n \setminus (\text{int } B \cup \Phi^{-1}(U))$  is closed, and we may conclude that  $\Phi$  is USC.

Finally, observe that if  $a \in \Phi(x)$  and  $-a \in \Phi(x)$ , then  $B \subset x + H_a$ , and hence  $B$  is not a convex body.  $\square$

We finish this section with one more definition and a lemma.

A continuous map  $f : X \rightarrow Y$  is called *proper* if the pre-image of every compactum in  $Y$  is compact. If  $A \subset X$ , then the restriction of  $f$  to  $A$  is denoted  $f|_A$ . Recall that in metric spaces a continuous map is proper if and only if it is closed and every fibre is compact; see Engelking [6, Theorem 3.7.18]. We will use the following observation concerning composition and proper maps.

**Lemma 6.** *If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then the following statements are equivalent:*

- (1)  $g \circ f : X \rightarrow Z$  is proper and
- (2) both  $f$  and  $g|_{f(X)} : f(X) \rightarrow Z$  are proper.

*Proof.* The implication (2)  $\Rightarrow$  (1) is trivial. Assume that  $g \circ f$  is proper. If  $C$  is a compactum in  $Y$ , then  $(g \circ f)^{-1}(g(C))$  is a compact subset of  $X$  that contains  $f^{-1}(C)$ ; thus  $f$  is proper. If  $C$  is a compactum in  $Z$ , then  $f((g \circ f)^{-1}(C))$  is a compact subset of  $Y$  that equals  $g^{-1}(C) \cap f(X)$ ; thus  $g|_{f(X)}$  is proper.  $\square$

### 3. THE LEMMAS

In this section we give a number of results about projection properties of closed convex sets.

**Lemma 7.** *Let  $0 < m < n$ , let  $B$  be a closed convex set in  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_m^n$ . Let  $x \in \mathbb{R}^n \setminus \text{int } B$  and suppose that all supporting hyperplanes to  $B$  at  $x$  are consistent with  $\mathcal{P}$ . Then there is a neighbourhood  $V$  of  $x$  in  $\mathbb{R}^n \setminus \text{int } B$  such that for every point  $y \in V$  any supporting hyperplane to  $B$  at  $y$  is consistent with  $\mathcal{P}$ .*

*Proof.* Consider

$$U = \{u \in S^{n-1} : u \text{ is perpendicular to some element of } \mathcal{P}\}.$$

First we show that  $U$  is open in  $S^{n-1}$ . Let  $u$  be in  $U$  and let  $L \in \mathcal{P}$  be such that  $L \subset H_u$ . Choose a basis  $b_1, \dots, b_m$  for  $L$  consisting of elements of  $S^{n-1}$ . Since  $\mathcal{P}$  is open we have by Lemma 2 that there exists a  $\delta > 0$  such that whenever  $F = \{b'_1, \dots, b'_m\} \subset \mathbb{R}^n$  has the property  $\|b'_i - b_i\| < \delta$  for every  $i$ , then  $[F] \in \mathcal{P}$ . Let  $v$  be an element of  $S^{n-1}$  such that  $\|v - u\| < \delta$ . We let  $T$  be the orthogonal transformation of  $\mathbb{R}^n$  that is generated by a rotation in the plane  $\{u, v\}$  that carries  $u$  to  $v$  and the identity on  $\{u, v\}^\perp$ . Then we have that  $\|b_i - Tb_i\| < \delta$  for each  $i \leq m$ , and hence  $[Tb_1, \dots, Tb_m]$  is an element of  $\mathcal{P}$  that is perpendicular to  $v$ . Thus  $v \in U$ , and  $U$  is open in  $S^{n-1}$ .

Since by assumption  $\Phi(x) \subset U$  we have by Lemma 5 that there is a neighbourhood  $V$  of  $x$  in  $\mathbb{R}^n \setminus \text{int } B$  such that  $\Phi(y) \subset U$  for each  $y \in V$ .  $\square$

**Lemma 8.** *Let  $0 < m < n$ , let  $B$  be a closed and convex set in  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_m^n$ . Suppose that  $x \in \mathbb{R}^n$  is such that there are two distinct supporting hyperplanes at  $x$  to  $B$ , one of which is consistent with  $\mathcal{P}$ . Then there are a supporting hyperplane  $H$  to  $B$  at  $x$ , a  $P \in \mathcal{P}$ , and a line  $\ell \subset P$  such that  $x + P \subset H$  and  $\psi_\ell|_B : B \rightarrow \mathbb{R}^n$  is proper (and hence  $\psi_\ell(B)$  is closed).*

*Proof.* Choose a coordinate system for  $\mathbb{R}^n$  such that  $x = \mathbf{0}$ . By assumption there are distinct supporting hyperplanes  $H_1, H_2$  to  $B$  at  $\mathbf{0}$ , and there are  $v_1, v_2 \in \Phi(\mathbf{0})$  such that  $v_1 \neq \pm v_2$ ,  $v_1 \perp H_1$ , and  $v_2 \perp H_2$ . Moreover, we may assume that there is a  $P_1 \in \mathcal{P}$  with  $P_1 \subset H_1$ . Since  $H_1 \neq H_2$  we have that  $L = H_1 \cap H_2$  is

$(n - 2)$ -dimensional. Since  $\dim(P_1 \cap L) = \dim(P_1 \cap H_2) \geq m - 1$  we can select a subspace  $P'$  of  $P_1 \cap L$  with  $\dim P' = m - 1$ . Select a basis  $\{e_1, \dots, e_m\}$  for  $P_1$  such that  $\{e_1, \dots, e_{m-1}\}$  is a basis for  $P'$ . With Lemma 2 we can find an  $\varepsilon > 0$  such that  $[\{e_1, \dots, e_{m-1}, u\}] \in \mathcal{P}$  for each  $u$  with  $\|e_m - u\| < \varepsilon$ . We can select a vector  $e'_m \in H_1 \setminus L$  such that  $\|e'_m - e_m\| < \varepsilon$ . Then obviously  $e'_m \cdot v_2 \neq 0$ , and we may assume that  $e'_m \cdot v_2 > 0$  because we may replace  $e_m$  and  $e'_m$  by their opposite vectors. Note that  $e'_m \cdot v_1 = 0$ . Using Lemma 2 in the same way as above, we can select an approximation  $e$  to  $e'_m$  such that  $P = [\{e_1, \dots, e_{m-1}, e\}] \in \mathcal{P}$ ,  $e \cdot v_2 > 0$ , and  $e \cdot v_1 < 0$ . Let  $\ell$  be the line  $\mathbb{R}e$  in  $P$ .

Let

$$a = \frac{v_2}{e \cdot v_2} - \frac{v_1}{e \cdot v_1}$$

and note that  $a \neq \mathbf{0}$  because  $v_1 \neq \pm v_2$  and that  $a \cdot e = 0$ . Note that  $H_a = L + \ell$ , so it contains  $P$ . If  $y \in B$ , then  $y \cdot v_1 \leq 0$  and  $y \cdot v_2 \leq 0$ , so also  $y \cdot a \leq 0$ , and hence  $a/\|a\| \in \Phi(\mathbf{0})$ , and  $H_a$  is a supporting hyperplane.

Let  $y \in B$  and let  $z = \psi_\ell(y)$  so  $y = z + \alpha e$  for some  $\alpha \in \mathbb{R}$ . Let  $j = 1, 2$  and note that  $v_j \cdot y \leq 0$  because  $v_j \in \Phi(\mathbf{0})$ . Then  $\alpha e \cdot v_j \leq -z \cdot v_j \leq \|z\|$ . Since  $e \cdot v_1 < 0$  and  $e \cdot v_2 > 0$  we have

$$\frac{\|z\|}{e \cdot v_1} \leq \alpha \leq \frac{\|z\|}{e \cdot v_2}.$$

Thus we see that the preimage under  $\psi_\ell \upharpoonright B$  of every bounded set is bounded, which means that the map is proper because  $B$  is closed.  $\square$

**Lemma 9.** *Let  $0 < m < n$ , let  $C$  and  $B$  be convex closed subsets of  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_m^n$ . If  $C$  is a weak  $\mathcal{P}$ -imitation of  $B$ , then  $C$  and  $B$  have precisely the same (derived)  $\mathcal{P}$ -faces, and hence  $\mathcal{E}_k(C, \mathcal{P}) = \mathcal{E}_k(B, \mathcal{P})$  for each  $k$ .*

*Proof.* It suffices to prove the result for  $\mathcal{P}$ -faces. Let  $H$  be a supporting hyperplane to  $C$  in  $\mathbb{R}^n$  that contains a  $P \in \mathcal{P}$ . Note that  $\psi_P(H)$  is then a supporting hyperplane to  $\psi_P(C)$  (and hence also to  $\overline{\psi_P(C)}$ ) in  $P^\perp$ . Since  $\overline{\psi_P(C)} = \overline{\psi_P(B)}$  we have that  $H$  is a supporting hyperplane to  $B$ .

Now consider a hyperplane  $H_1$  that supports both  $B$  and  $C$  and that is consistent with  $\mathcal{P}$  such that  $H_1 \cap B \neq H_1 \cap C$ . By symmetry we may assume that there is an  $x \in H_1 \cap C \setminus B$ . Choose a coordinate system such that  $x = \mathbf{0}$ . Let  $H_2$  be the (unique) hyperplane through  $\mathbf{0}$  with  $\varepsilon = d(H_2, B) = d(\mathbf{0}, B) > 0$ ; see [9, p. 347]. Let  $P_1 \in \mathcal{P}$  be such that  $P_1 \subset H_1$ . Since  $\psi_{P_1}(\mathbf{0}) \in \psi_{P_1}(C) \subset \overline{\psi_{P_1}(B)}$  we have that  $d(P_1, B) = 0$ , and hence  $d(H_1, B) = 0$ . Thus we have that  $H_1 \neq H_2$ , and we can find a supporting hyperplane  $H_a$  at  $\mathbf{0}$  and a  $P \in \mathcal{P}$  with  $P \subset H_a$ , precisely as in the proof of Lemma 8. Let  $y \in B$  and note that  $y \cdot v_1 \leq 0$  and  $y \cdot v_2 = -d(y, H_2) \leq -\varepsilon$ . Thus we have that  $y \cdot a \leq -\varepsilon/e \cdot v_2$ , and hence  $-y \cdot (a/\|a\|) = d(y, H_a) \geq \frac{\varepsilon}{e \cdot v_2 \|a\|}$ . Since  $P \subset H_a$  we now have that  $d(\psi_P(B), \psi_P(H_a)) = d(B, H_a) \geq \frac{\varepsilon}{e \cdot v_2 \|a\|}$ . Thus  $\psi_P(\mathbf{0}) \in \psi_P(C) \setminus \overline{\psi_P(B)}$ , which contradicts the premise that  $C$  is a weak  $\mathcal{P}$ -imitation of  $B$ . The proof is complete.  $\square$

**Lemma 10.** *Let  $0 < m < n$ , let  $B$  be a convex body in  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be an open set in  $\mathbb{L}_m^n$ . If there is a  $P \in \mathcal{P}$  such that  $\psi_P(B) \neq P^\perp$ , then there is a  $P' \in \mathcal{P}$  such that  $\psi_{P'}(B)$  is closed and  $\psi_{P'}(B) \neq (P')^\perp$ .*

*Proof.* Define

$$\mathcal{A} = \{L : L \text{ is a linear subspace of some } P' \in \mathcal{P} \text{ such that } \psi_L(B) \text{ is closed and } \psi_{P'}(B) \neq (P')^\perp\}.$$

Note that  $\psi_{\{0\}}(B) = B$  is closed and  $\{0\} \subset P$ ; thus  $\{0\} \in \mathcal{A}$ . We may define

$$l = \max\{\dim L : L \in \mathcal{A}\}.$$

It suffices to show that  $l = m$ , so let us assume that  $l < m$ . Choose linear subspaces  $P_1 \in \mathcal{P}$  and  $L \subset P_1$  such that  $\dim L = l$ ,  $\psi_L(B)$  is closed, and  $\psi_{P_1}(B) \neq (P_1)^\perp$ . Define

$$B_L = \psi_L(B) \quad \text{and} \quad \mathcal{P}_L = \{N \in \mathbb{L}_{m-l}(L^\perp) : N + L \in \mathcal{P}\}.$$

Clearly,  $N = \psi_L(P_1) \in \mathcal{P}_L$ . By Corollary 4,  $\mathcal{P}_L$  is open in  $\mathbb{L}_{m-l}(L^\perp)$ . We have that  $\psi_N(B_L) = \psi_{P_1}(B)$  is not closed because  $l < m$ . Select an  $x \in \overline{\psi_N(B_L)} \setminus \psi_N(B_L)$ . By the Hahn-Banach Theorem there exists a supporting hyperplane  $H$  in  $(P_1)^\perp$  at  $x$  to  $\psi_N(B_L)$ . Put  $H_1 = H + N$  and note that  $H_1$  is a hyperplane in  $L^\perp$  that supports  $B_L$  at  $x$ . Observe that  $x \notin B_L$  and  $B_L$  is closed and convex, so we can find a hyperplane  $H_2$  through  $x$  such that  $d(H_2, B_L) = d(x, B_L) > 0$ ; see [9, p. 347]. Now, note that  $H_1 \neq H_2$  since  $d(H_1, B_L) = 0$ . Consequently, there are at least two supporting hyperplanes at  $x$  to  $B_L$  in  $L^\perp$ . Next, in  $L^\perp$  we can apply Lemma 8 to  $B_L$ ,  $\mathcal{P}_L$ , and  $x$  to get an  $N' \in \mathcal{P}_L$  and a line  $\ell \subset N'$  such that  $\psi_\ell(B_L)$  is closed. Moreover, there is a hyperplane  $V$  in  $L^\perp$  that contains  $N'$  and does not cut  $B_L$ . Set

$$L' = L + \ell \quad \text{and} \quad P'_1 = L + N'.$$

Since  $N' \subset V$  we find that  $\psi_{N'}(B_L)$  is contained in a half-space of  $(P'_1)^\perp$ , and hence  $\psi_{P'_1}(B) = \psi_{N'}(B_L) \neq (P'_1)^\perp$ . We have that  $\psi_{L'}(B) = \psi_\ell(B_L)$  is closed. We now have that  $L' \in \mathcal{A}$  and  $\dim L' = l + 1$ , which contradicts the maximality of  $l$ . The proof is complete.  $\square$

**Lemma 11.** *Let  $0 < m < n$ , let  $B$  be closed and convex in  $\mathbb{R}^n$  such that  $\dim B \geq n - m$ , and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_m^n$ . If  $B$  is not a  $\mathcal{P}$ -imitation of  $\text{aff } B$ , then there are a  $P \in \mathcal{P}$  and a linear subspace  $L$  of  $P$  such that*

- (a)  $\psi_L \upharpoonright B$  is a proper map and  $\psi_L(\text{aff } B) = L^\perp$ ,
- (b)  $\psi_P(B) \neq P^\perp$  and  $\psi_P(B)$  is closed, and
- (c) *if  $\dim L < m$  and if  $H$  is a supporting hyperplane at some  $w \in \partial(\psi_L(B))$  to  $\psi_L(B)$  in  $L^\perp$  such that  $H + L$  is consistent with  $\mathcal{P}$ , then  $H$  is the only supporting hyperplane at  $w$  to  $\psi_L(B)$  in  $L^\perp$ .*

*Proof.* Let  $M = \text{aff } B$  and choose a coordinate system such that  $0 \in M$ . Let  $k = n - \dim B$  and note that  $0 \leq k \leq m$ . Define

$$\begin{aligned} \mathcal{A} = \{L : L \text{ is a linear subspace of some } P \in \mathcal{P} \text{ such that} \\ \text{(a) and (b) hold and such that } L \text{ contains} \\ \text{a } k\text{-subspace } E \text{ with } E \cap M = \{0\}\}. \end{aligned}$$

Since  $B$  is not a  $\mathcal{P}$ -imitation of  $\text{aff } B$  there is a  $P^* \in \mathcal{P}$  such that  $\psi_{P^*}(B) \neq \psi_{P^*}(\text{aff } B)$ .

*Claim 1.* We may assume that  $P^*$  contains a  $k$ -subspace  $E$  with  $E \cap M = \{0\}$ .



*Proof.* We consider two cases.

Case I:  $\dim(M \cap P^*) \leq m - k$ . Then there is room to find a  $k$ -subspace  $E$  in  $P^*$  with  $E \cap M = \{\mathbf{0}\}$ .

Case II:  $\dim(M \cap P^*) > m - k$ . Let  $P_1$  be an  $(m - k)$ -subspace of  $M \cap P^*$  and put  $E_1 = \psi_{P_1}(P^*)$  so  $P^* = P_1 + E_1$ . Note that

$$\psi_{P_1}(B) \subsetneq \psi_{P_1}(M) \subsetneq M.$$

Since  $\dim E_1 + \dim M = k + \dim B = n$  we can use Lemma 2 to rotate  $E_1$  slightly to find a  $k$ -subspace  $E$  of  $\mathbb{R}^n$  such that

$$P_1 + E \in \mathcal{P} \quad \text{and} \quad E \cap M = \{\mathbf{0}\}.$$

Note that  $\psi_E \upharpoonright M$  is a one-to-one map. Put  $P_1^* = P_1 + E$  and observe that

$$\psi_{P_1^*}(B) = \psi_E(\psi_{P_1}(B)) \neq \psi_E(\psi_{P_1}(M)) = \psi_{P_1^*}(M).$$

Thus in this case we may use  $P_1^*$  to replace  $P^*$ .  $\square$

*Claim 2.*  $E \in \mathcal{A}$ .

*Proof.* If  $k = m$ , then  $E = P^*$  and  $\psi_{P^*} \upharpoonright M: M \rightarrow (P^*)^\perp$  is an isomorphism. This means that  $\psi_{P^*}(B)$  is closed in  $\mathbb{R}^n$ . Now assume that  $k < m$ . Let  $P' = \psi_E(P^*)$ ,  $B' = \psi_E(B)$ , and

$$\mathcal{P}_E = \{F \in \mathbb{L}_{m-k}(N^\perp) : F + E \in \mathcal{P}\}.$$

Since  $E \cap M = \{\mathbf{0}\}$  the map  $\psi_E \upharpoonright M$  is an isomorphism between  $M$  and  $\psi_E(M) = E^\perp$ . Note that  $B'$  is a convex body in  $E^\perp$  and that  $P' \in \mathcal{P}_E$  with  $\psi_{P'}(B') = \psi_{P^*}(B) \neq \psi_{P^*}(M) = (P^*)^\perp$ . Since  $\mathcal{P}_E$  is open by Corollary 4 we may assume according to Lemma 10 that  $\psi_{P'}(B') = \psi_{P^*}(B)$  is closed. Noting that  $\psi_E \upharpoonright M$  is proper because it is an isomorphism we find that  $E \in \mathcal{A}$ .  $\square$

We may now define

$$l = \max\{\dim L : L \in \mathcal{A}\}.$$

Select linear spaces  $P \in \mathcal{P}$ ,  $L \subset P$ , and  $F \subset L$  such that  $\dim L = l$ ,  $\dim F = k$ ,  $F \cap M = \{\mathbf{0}\}$ , and conditions (a) and (b) are satisfied. Put  $B_L = \psi_L(B)$ . We show that  $L$  satisfies condition (c). Striving for a contradiction, let us assume that (c) does not hold. Then  $l < m$ , and there is a point  $w \in \partial B_L$  such that there are two distinct supporting hyperplanes  $H_1$  and  $H_2$  at  $w$  to  $B_L$  in  $L^\perp$  with  $H_1 + L$  consistent with  $\mathcal{P}$ . Since  $F$  and  $M$  are complementary spaces in  $\mathbb{R}^n$ , we have that  $\psi_F(B)$  is a convex body in  $F^\perp$  and hence  $B_L$  is a convex body in  $L^\perp$ . Define

$$\mathcal{P}_L = \{N \in \mathbb{L}_{m-l}(L^\perp) : N + L \in \mathcal{P}\}$$

and note that  $\psi_L(P) \in \mathcal{P}_L$  and that  $\mathcal{P}_L$  is open in  $\mathbb{L}_{m-l}(L^\perp)$  by Corollary 4. We may now apply Lemma 8 to  $w$ ,  $B_L$ , and  $\mathcal{P}_L$  in  $L^\perp$  to get an  $N \in \mathcal{P}_L$  and a line  $\ell \subset N$  such that  $\psi_\ell \upharpoonright B_L$  is proper. Moreover, there is a supporting hyperplane  $V$  at  $w$  to  $B_L$  in  $L^\perp$  such that  $N \subset V$ . Put  $P_1 = L + N \in \mathcal{P}$  and  $L' = L + \ell \subset P_1$ .

*Claim 3.*  $L' \in \mathcal{A}$ .

*Proof.* Since  $N \subset V$  we find that  $\psi_N(B_L)$  is contained in a half-space of  $(P_1)^\perp$ , and hence  $\psi_{P_1}(B) = \psi_N(B_L) \neq (P_1)^\perp$ . Since  $\psi_{L'} \upharpoonright B = (\psi_\ell \upharpoonright B_L) \circ (\psi_L \upharpoonright B)$  it is a proper map and  $B_{L'} = \psi_{L'}(B)$  is closed.

If  $l = m - 1$ , then  $\ell = N$  and  $L' = P_1$ . Consequently,  $L' \in \mathcal{A}$ .

Now let  $l \leq m - 2$ . Set

$$\mathcal{P}_{L'} = \{T \in \mathbb{L}_{m-l-1}((L')^\perp) : T + L' \in \mathcal{P}\}.$$

Let  $N_1 = \psi_\ell(N)$  and note that  $N_1 \in \mathcal{P}_{L'}$  because  $P_1 = N_1 + L'$ . Since  $\psi_{N_1}(B_{L'}) = \psi_{P_1}(B) \neq (P_1)^\perp$  we may apply Lemma 10 to  $B_{L'}$  and  $\mathcal{P}_{L'}$  in  $(L')^\perp$  to obtain an  $N'_1 \in \mathcal{P}_{L'}$  such that  $\psi_{N'_1}(B_{L'}) = \psi_{N'_1+L'}(B)$  is a closed and proper subset of  $(N'_1 + L')^\perp$ . Note that  $N'_1 + L' \in \mathcal{P}$  and hence  $L' \in \mathcal{A}$ .  $\square$

Since  $\dim L' = l + 1$ , Claim 3 violates the maximality of  $l$ . The proof is complete.  $\square$

*Remark 3.* If we replace the premise of Lemma 11 that  $B$  is not a  $\mathcal{P}$ -imitation of  $\text{aff } B$  by the condition  $\mathcal{P} \neq \emptyset$ , then by the same proof there still are a  $P \in \mathcal{P}$  and a linear subspace  $L$  of  $P$  that satisfy (a) and (c).

**Lemma 12.** *Let  $0 < m < n$ , let  $B$  be closed and convex in  $\mathbb{R}^n$  such that  $\dim B \geq n - m$ , and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_m^n$ . If  $B$  is not a  $\mathcal{P}$ -imitation of  $\text{aff } B$ , then there is a linear subspace  $L$  of  $\mathbb{R}^n$  such that  $\psi_L|_B$  is a proper map,  $\psi_L(\text{aff } B) = L^\perp$ , and there is a  $w \in \partial\psi_L(B)$  such that every supporting hyperplane  $H$  at  $w$  to  $\psi_L(B)$  in  $L^\perp$  is consistent with  $\mathcal{P}_L$ .*

*Proof.* Choose a  $P \in \mathcal{P}$  and an  $L \subset P$  that satisfy the properties (a), (b), and (c) of Lemma 11. Since  $\psi_L(\text{aff } B) = L^\perp$  and  $L \subset P$  we have that  $\text{aff}(\psi_P(B)) = \psi_P(\text{aff } B) = P^\perp$ . Thus with property (b) we can find an  $x \in B$  such that  $\psi_P(x) \in \partial\psi_P(B)$ . Note that  $w = \psi_L(x)$  is an element of  $\partial\psi_L(B)$ . If  $\dim L = m$ , then  $L = P$ , and every hyperplane  $H$  in  $L^\perp$  is trivially consistent with  $\mathcal{P}_L = \{\{\mathbf{0}\}\}$ . Now let  $\dim L < m$ . Select a supporting hyperplane  $H_1$  to  $\psi_P(B)$  at  $\psi_P(x)$  in  $P^\perp$ . Put  $P' = \psi_L(P)$  and  $H'_1 = H_1 + P'$ . Note that  $H'_1$  is a supporting hyperplane to  $\psi_L(B)$  at  $w$  such that  $H'_1 + L = H_1 + P \in \mathcal{P}$ , and hence  $H'_1 \in \mathcal{P}_L$ . By property (c) we now have that every supporting hyperplane to  $\psi_L(B)$  at  $w$  in  $L^\perp$  is equal to  $H'_1$ , and hence an element of  $\mathcal{P}_L$ .  $\square$

**Lemma 13.** *Let  $0 < m < n$ , let  $B$  be a closed subset of  $\mathbb{R}^n$  such that  $\dim(\text{aff } B) < n - m$ , and let  $\mathcal{P}$  be a nonempty open subset of  $\mathbb{L}_m^n$ . Then  $B$  has only one closed weak  $\mathcal{P}$ -imitation.*

*Proof.* Let  $C$  be a closed weak  $\mathcal{P}$ -imitation of  $B$ . Let  $x \in C$  be arbitrary and put  $L = \text{aff}(B \cup \{x\})$ . Since  $\dim L \leq n - m$  we can find with Lemma 2 a  $P \in \mathcal{P}$  such that  $P \cap (L - y) = \{\mathbf{0}\}$  for each  $y \in L$ . Then  $\psi_P|_L: L \rightarrow \psi_P(L)$  is an isomorphism and  $\psi_P(x) \in \psi_P(C) \subset \overline{\psi_P(B)} = \psi_P(B)$ ; thus  $x \in B$ . So  $C \subset B$ . Let  $P$  be one of the elements of  $\mathcal{P}$  such that  $\psi_P|_{\text{aff } B}$  is an isomorphism. Since  $\overline{\psi_P(C)} = \overline{\psi_P(B)}$  we have that  $C = B$ .  $\square$

**Lemma 14 (Lifting Lemma).** *Let  $B$  be a convex closed subset of  $\mathbb{R}^n$ , let  $L$  be a linear subspace of  $\mathbb{R}^n$ , and let  $A \subset \psi_L(B)$ . If  $(w + L) \cap B$  is a singleton for every point  $w \in A$ , then there is a continuous function  $f: A \rightarrow B$  such that  $\psi_L \circ f$  is the identity map on  $A$ ; that is,  $A$  can be lifted to a homeomorphic subset  $f(A)$  of  $B$ .*

*Proof.* Let  $f: A \rightarrow B$  be the function that is defined by

$$\{f(w)\} = (w + L) \cap B.$$

We need to show that  $f$  is continuous. Let  $w \in A$  be such that there are an  $\varepsilon > 0$  and a sequence  $w_1, w_2, \dots$  in  $A$  with  $\lim_{i \rightarrow \infty} w_i = w$  and  $\|f(w_i) - f(w)\| \geq \varepsilon$  for

every  $i \in \mathbb{N}$ . Since  $S^{n-1}$  is compact we can find a subsequence  $w_{i_1}, w_{i_2}, \dots$  such that

$$\lim_{j \rightarrow \infty} \frac{f(w_{i_j}) - f(w)}{\|f(w_{i_j}) - f(w)\|} = u$$

for some  $u \in S^{n-1}$ . Put  $t_j = \varepsilon / \|f(w_{i_j}) - f(w)\|$  and note that  $t_j \in [0, 1]$  for  $j \in \mathbb{N}$ . By convexity we have  $f(w) + t_j(f(w_{i_j}) - f(w)) \in B$  for  $j \in \mathbb{N}$ . Note that  $\lim_{j \rightarrow \infty} (f(w) + t_j(f(w_{i_j}) - f(w))) = f(w) + \varepsilon u \in B$  because  $B$  is closed. Also we have

$$\begin{aligned} \psi_L(f(w) + \varepsilon u) &= \lim_{j \rightarrow \infty} \psi_L(f(w) + t_j(f(w_{i_j}) - f(w))) \\ &= \lim_{j \rightarrow \infty} (w + t_j(w_{i_j} - w)) = w. \end{aligned}$$

Thus  $f(w)$  and  $f(w) + \varepsilon u$  are distinct elements of  $(w + L) \cap B$  in contradiction to the premise.  $\square$

#### 4. THE MAIN THEOREMS

In this section we prove our main results. We start with stating a controlled version of the Tipping Lemma in [1].

**Lemma 15** (Tipping Lemma). *Let  $B$  be a closed convex set in  $\mathbb{R}^m$  for  $m \geq 2$ , let  $C$  be a closed subset of  $B$ , and let  $H$  be a hyperplane of  $\mathbb{R}^m$  that does not cut  $B$ . Suppose that  $V$  is a halfspace of  $H$  such that  $B \cap \partial V \neq \emptyset$ ,  $V \cap C = \emptyset$ , and  $V \cap B$  is bounded. If  $\varepsilon > 0$ , then there exists a halfspace  $V'$  of  $\mathbb{R}^m$  such that  $\partial V \subset \partial V'$ ,  $V \subset V'$ ,  $V' \cap C = \emptyset$ ,  $V' \cap B$  is bounded, and  $\rho(\partial V', H) < \varepsilon$ .*

The proof of this lemma is the same as the proof of [1, Lemma 2]. The only thing we need to note is that we obtain  $\partial V'$  in the proof of [1, Lemma 2] by rotating  $H$  about  $\partial V$  and that the angle of rotation can be made arbitrarily small.

Lemma 15 is needed in the following theorem which generalizes [1, Theorem 3].

**Theorem 16.** *Let  $0 < k < n$ , let  $B$  be a convex and closed set in  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be open in  $\mathbb{L}_{n-k}^n$ . If  $C$  is a closed set that is a weak  $\mathcal{P}$ -imitation of  $B$ , then  $\mathcal{E}_k(B, \mathcal{P}) \subset C$ .*

*Proof.* Let  $C$  be a closed set in  $\mathbb{R}^n$  such that for every  $P \in \mathcal{P}$ ,

$$\overline{\psi_P(C)} = \overline{\psi_P(B)}.$$

Consider the closed convex set  $\overline{\langle C \rangle}$  and note that  $C$ ,  $B$ , and  $\overline{\langle C \rangle}$  are all weak  $\mathcal{P}$ -imitations of each other because

$$\overline{\psi_P(C)} \subset \overline{\psi_P(\overline{\langle C \rangle})} \subset \overline{\langle \psi_P(C) \rangle} \subset \overline{\langle \psi_P(B) \rangle} = \overline{\psi_P(B)} = \overline{\psi_P(C)}$$

for each  $P \in \mathcal{P}$ . According to Lemma 9 we have  $\mathcal{E}_k(B, \mathcal{P}) = \mathcal{E}_k(\overline{\langle C \rangle}, \mathcal{P})$ . By replacing  $B$  by  $\overline{\langle C \rangle}$  we assume that  $C \subset B$ .

In order to prove that  $\mathcal{E}_k(B, \mathcal{P}) \subset C$  it suffices to show that every derived  $\mathcal{P}$ -face of  $B$  with dimension less than  $k$  is contained in  $C$  because  $C$  is closed. Assume that  $F$  is such a derived face of  $B$ . Striving for a contradiction we suppose that  $F \setminus C \neq \emptyset$ . Choose a rectangular coordinate system for  $\mathbb{R}^n$  such that  $\mathbf{0} \in F \setminus C$ . Since  $F$  is a derived  $\mathcal{P}$ -face of  $B$  we can find a sequence of affine spaces

$$\text{aff } F \subset H_{k-1} \subset H_k \subset \dots \subset H_{n-1} \subset H_n = \mathbb{R}^n$$

such that  $H_{n-1}$  is consistent with  $\mathcal{P}$ ,  $\dim H_i = i$  for each  $i$ , and  $H_i$  is a hyperplane in  $H_{i+1}$  that does not cut  $B \cap H_{i+1}$  for  $i \in \{k-1, \dots, n-1\}$ .

Let  $P^* \in \mathcal{P}$  be such that  $P^* \subset H_{n-1}$ . We have

$$\dim(H_{i+k-1} \cap P^*) \geq \dim H_{i+k-1} + \dim P^* - \dim H_{n-1} = i$$

for every  $i \in \{1, \dots, n-k\}$ . Let  $e_1^*, e_2^*, \dots, e_{n-k}^*$  be a basis for  $P^*$  such that  $e_i^* \in H_{i+k-1}$ . With Lemma 2 we find an  $\varepsilon > 0$  such that  $[\{e_1, \dots, e_{n-k}\}] \in \mathcal{P}$  whenever  $\|e_i - e_i^*\| < \varepsilon$  for  $1 \leq i \leq n-k$ . Select for every  $i \in \{1, \dots, n-k\}$  an  $e_i \in H_{i+k-1} \setminus H_{i+k-2}$  such that  $\|e_i - e_i^*\| < \varepsilon$ . Define  $P = [\{e_1, \dots, e_{n-k}\}]$  and note that  $P \in \mathcal{P}$  and  $P \subset H_{n-1}$ . We let  $e_{n-k+1}$  be a vector in  $H_n \setminus H_{n-1}$ .

We construct by induction a sequence  $V_1 \subset V_2 \subset \dots \subset V_{n-k+1}$  such that for  $1 \leq i \leq n-k+1$ :

- (1)  $V_i$  is a halfspace of  $[\{e_1, \dots, e_i\}]$ ,
- (2)  $\mathbf{0} \in \partial V_i$ ,
- (3)  $V_i \cap C = \emptyset$ , and
- (4)  $V_i \cap B$  is bounded.

Note that  $e_1 \in H_k \setminus H_{k-1}$  and that  $H_{k-1}$  does not cut  $B \cap H_k$ . For the base step we let  $V_1 \subset \mathbb{R}e_1$  be the ray that emanates from  $\mathbf{0}$  into the side of  $H_{k-1}$  that is disjoint from  $B$ . Note that  $B \cap V_1 = \{\mathbf{0}\} = \partial V_1$  and that the induction hypotheses are satisfied.

Now let  $1 \leq i \leq n-k$  and assume that  $V_i$  has been found. Let  $M = [\{e_1, \dots, e_{i+1}\}] \subset H_{k+i}$ . Put  $H = [\{e_1, \dots, e_i\}]$ ,  $C' = C \cap M$ , and  $B' = B \cap M$ . Since  $H = H_{k+i-1} \cap M$  we have that  $H$  does not cut  $B'$  in  $M$ . Apply Lemma 15 to  $H$ ,  $C'$ ,  $B'$ , and  $V_i$  with  $M$  as ambient space. We obtain a halfspace  $V_{i+1}$  of  $M$  such that  $\partial V_i \subset \partial V_{i+1}$ ,  $V_i \subset V_{i+1}$ ,  $V_{i+1} \cap C' = V_{i+1} \cap C = \emptyset$ , and  $V_{i+1} \cap B' = V_{i+1} \cap B$  is bounded. The induction hypotheses for  $i+1$  are satisfied. We now look at the step  $i = n-k$  more closely. In that case  $H = P$ , so we may assume that  $\partial V_{n-k+1}$  is close enough to  $P$  so that  $\partial V_{n-k+1} \in \mathcal{P}$  as well.

Let  $N = \partial V_{n-k+1}$ . Since  $N \cap C = \emptyset$  and  $N \cap B$  is bounded we can now show by precisely the same method as in the proof of [1, Theorem 3] that  $\psi_N(\mathbf{0}) \notin \overline{\psi_N(C)}$ . Since  $\mathbf{0} \in B$  we have that  $C$  is not a weak  $\mathcal{P}$ -imitation of  $B$ , and the proof is complete.  $\square$

*Remark 4.* The boundary of a convex body in  $\mathbb{R}^k$  is homeomorphic to either

- (1)  $\emptyset$ , or
- (2)  $\mathbb{R}^{k-1}$ , or
- (3)  $S^{i-1} \times \mathbb{R}^{k-i}$  for some  $i \in \{1, 2, \dots, k\}$ ;

cf. [1, Remark 1].

With the following theorem we put it all together.

**Theorem 17.** *Let  $0 < k < n$ , let  $B$  be a convex and closed subset of  $\mathbb{R}^n$  with  $\dim B \geq k$ , and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_{n-k}^n$ . Then the following statements are equivalent:*

- (1)  $B$  contains no  $k$ -plane, and  $B$  is not a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ .
- (2)  $B$  contains no  $k$ -plane, and  $B$  is not a  $\mathcal{P}$ -imitation of  $\text{aff } B$ .
- (3) There is a nonempty closed subset  $A$  of  $\mathcal{E}_k(B, \mathcal{P})$  that is homeomorphic to the boundary of a convex body in  $\mathbb{R}^k$ .
- (4)  $\mathcal{E}_k(B, \mathcal{P}) \neq \emptyset$ .

*Proof.* The implication (3)  $\Rightarrow$  (4) requires no proof.

We show that (4)  $\Rightarrow$  (1). If  $B$  is a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ , then  $\mathcal{E}_k(B, \mathcal{P}) = \mathcal{E}_k(\mathbb{R}^n, \mathcal{P}) = \emptyset$  by Lemma 9. Suppose now that  $B$  contains a  $k$ -plane. According to [1, Lemma 4] this means that every nonempty derived face of  $B$  contains a  $k$ -plane. Since every derived  $\mathcal{P}$ -face is a derived face we again have  $\mathcal{E}_k(B, \mathcal{P}) = \emptyset$ , which proves the point.

We turn to proving the implication (1)  $\Rightarrow$  (2). Assume that (1) is valid and that  $B$  is a  $\mathcal{P}$ -imitation of  $\text{aff } B$ . Since  $B$  is not a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$  we can find a hyperplane  $H$  in  $\mathbb{R}^n$  that does not cut  $B$  and that contains  $x + P$  for some  $P \in \mathcal{P}$  and  $x \in H$ . Since  $\psi_P(\text{aff } B) = \psi_P(B)$  we have  $\psi_H(\text{aff } B) = \psi_H(B) \neq H^\perp$ . Thus  $\psi_H(\text{aff } B)$  is an affine space that is a proper subset of the line  $H^\perp$ , and hence  $\psi_H(\text{aff } B) \subset \{a\}$  for some  $a \in H^\perp$ . So  $\text{aff } B$  is contained in the hyperplane  $\psi_H^{-1}(a)$  which contains  $a + P$ . We have that  $\text{aff } B$  is a  $\mathcal{P}$ -face of itself, and hence  $\text{aff } B$  is a  $\mathcal{P}$ -face of  $B$  by Lemma 9. Thus  $B = \text{aff } B$ . Since  $\dim B \geq k$  we have that  $\text{aff } B$  contains a  $k$ -plane which violates property (1).

To prove (2)  $\Rightarrow$  (3) we assume property (2).

*Claim 4.* Without loss of generality we may assume that every  $\mathcal{P}$ -face of  $B$  is contained in  $\mathcal{E}_k(B, \mathcal{P})$ .

*Proof.* Consider the collection  $\mathcal{D}$  consisting of all derived  $\mathcal{P}$ -faces of  $B$  whose dimension is at least  $k$ . Suppose that  $\mathcal{D} \neq \emptyset$  and select an  $F \in \mathcal{D}$  with minimal dimension. By the definition of  $\mathcal{D}$  all the faces of  $F$  have dimension less than  $k$ , and hence they are contained in  $\mathcal{E}_k(B, \mathcal{P})$ . Since every point of  $\partial F$  is contained in some face of  $F$  we get  $\partial F \subset \mathcal{E}_k(B, \mathcal{P})$ . Select a  $k$ -plane  $M$  in  $\text{aff } F$  such that the closed convex set  $G = F \cap M$  is  $k$ -dimensional. If  $\partial G = \emptyset$ , then  $G = M$  and  $B$  contains a  $k$ -plane. Thus  $\partial G \neq \emptyset$  and property (3) is proved because  $\partial G \subset \partial F \subset \mathcal{E}_k(B, \mathcal{P})$ . We may now assume that  $\mathcal{D} = \emptyset$ , which means that whenever  $H$  is a supporting hyperplane to  $B$  consistent with  $\mathcal{P}$ , then  $\dim(H \cap B) < k$  and  $H \cap B \subset \mathcal{E}_k(B, \mathcal{P})$ .  $\square$

Define

$$\begin{aligned} \mathcal{A} = \{L : L \text{ is a linear subspace of } \mathbb{R}^n \text{ such that } \psi_L|_B \text{ is proper,} \\ \psi_L(\text{aff } B) = L^\perp, \text{ and there is a } w \in \partial\psi_L(B) \text{ such that} \\ \text{every supporting hyperplane } H \text{ at } w \text{ to } \psi_L(B) \text{ in } L^\perp \\ \text{is consistent with } \mathcal{P}_L\}. \end{aligned}$$

According to Lemma 12 we have  $\mathcal{A} \neq \emptyset$ , so we may define

$$l = \min\{\dim L : L \in \mathcal{A}\}.$$

Choose an  $L \in \mathcal{A}$  such that  $\dim L = l$ . We put  $B_L = \psi_L(B)$  and note that  $B_L$  is a convex body in  $L^\perp$  because  $\text{aff } B_L = \psi_L(\text{aff } B) = L^\perp$ . We define

$$\begin{aligned} U = \{w \in \partial B_L : \text{every supporting hyperplane } H \text{ at } w \text{ to } B_L \\ \text{in } L^\perp \text{ is consistent with } \mathcal{P}_L\} \end{aligned}$$

and note that  $U$  is a nonempty set that is open in  $\partial B_L$  by Lemma 7.

*Claim 5.*  $B \cap (w + L)$  is a singleton for every  $w \in U$ .

*Proof.* Let  $w \in U \subset \partial B_L$  be arbitrary. Since  $\psi_L|_B$  is proper we have that  $B_L$  is closed and hence  $B \cap (w + L) \neq \emptyset$ .

Assume now that  $B \cap (w + L)$  contains two distinct points  $x$  and  $y$ . Then  $l \geq 1$  and  $y - x \in L$ , and we let  $\ell$  be the line in  $L$  through  $\mathbf{0}$  and  $y - x$ . We put  $L' = \psi_\ell(L)$ , and we will show that  $L' \in \mathcal{A}$ . We have by Lemma 6 that  $\psi_{L'} \upharpoonright B$  is proper because  $\psi_L \upharpoonright B$  is proper and  $\psi_L = \psi_\ell \circ \psi_{L'}$ . Note that  $\psi_L(\text{aff } B) = L^\perp$  is equivalent to  $L + \text{aff } B = \mathbb{R}^n$ , and hence  $\mathbb{R}^n = L' + \ell + \text{aff } B = L + \text{aff } B$  and  $\psi_{L'}(\text{aff } B) = (L')^\perp$ . Define  $z = (x + y)/2 \in B \cap (w + L)$  and note that  $w' = \psi_{L'}(z) \in \partial\psi_{L'}(B)$ . Let  $H'$  be a supporting hyperplane at  $w'$  to  $\psi_{L'}(B)$  in  $(L')^\perp$  and let  $u \in (L')^\perp$  be a vector such that  $H' = \{v \in (L')^\perp : (v - w') \cdot u = 0\}$  and  $(v - w') \cdot u \leq 0$  for all  $v \in \psi_{L'}(B)$ . So we have that  $(\psi_{L'}(y) - w') \cdot u = \frac{1}{2}\psi_{L'}(y - x) \cdot u \leq 0$  and  $(\psi_{L'}(x) - w') \cdot u = \frac{1}{2}\psi_{L'}(x - y) \cdot u \leq 0$ . Consequently,  $\psi_{L'}(y - x) \cdot u = (y - x) \cdot u = 0$  and hence  $w' + \ell \subset H'$ . Put  $H = \psi_\ell(H')$  and note that it is a supporting hyperplane to  $B_L$  at  $w$  in  $L^\perp$ . Thus by the definition of  $\mathcal{A}$  we have that  $H$  contains some  $P \in \mathcal{P}_L$ . Thus  $P + \ell \subset H + \ell = H'$  and  $P + \ell + L' = P + L \in \mathcal{P}$ . We have shown that  $H'$  is consistent with  $\mathcal{P}_{L'}$  and consequently  $L' \in \mathcal{A}$ . Since  $\dim L' = l - 1$  this result violates the minimality of  $l$ .  $\square$

If we combine Claim 5 with Lemma 14 we find a continuous function  $f: U \rightarrow B$  such that  $\psi_L \circ f$  is the identity. We show that  $f(U)$ , which is homeomorphic to  $U$ , is contained in  $\mathcal{E}_k(B, \mathcal{P})$ . Let  $w \in U \subset \partial B_L$  and select a supporting hyperplane  $H$  to  $B_L$  at  $w$  in  $L^\perp$ . Then  $H + L$  is a supporting hyperplane to  $B$  at  $f(w)$  in  $\mathbb{R}^n$ . Since  $L \in \mathcal{A}$  we have that  $H + L$  is consistent with  $\mathcal{P}$ , so by Claim 4 we have  $f(w) \in B \cap (H + L) \subset \mathcal{E}_k(B, \mathcal{P})$ .

By the definition of  $\mathcal{A}$  we have that  $\mathcal{P}_L \neq \emptyset$ , so there is a  $P \in \mathcal{P}$  that contains  $L$  and  $l \leq n - k$ . Since  $B_L$  is a convex body in  $L^\perp$  we have  $\dim B_L = \dim L^\perp = n - l \geq k$ . If  $\dim B_L = k$ , then  $L = P$  and hence  $U = \partial B_L$ . So  $\mathcal{E}_k(B, \mathcal{P})$  contains with  $f(U) = B \cap \psi_L^{-1}(\partial B_L)$  a closed copy of the boundary of the  $k$ -dimensional convex set  $B_L$ . If  $\dim B_L > k$ , then  $\partial B_L$  is a topological manifold with dimension at least  $k$ ; see Remark 4. Since  $U$  is a nonempty open subset of  $\partial B_L$  it contains a topological copy of  $S^{k-1}$ . Thus also  $f(U)$  and  $\mathcal{E}_k(B, \mathcal{P})$  contain a  $(k - 1)$ -sphere. The proof is complete.  $\square$

The following theorem is our main result from which Theorem 1 easily follows.

**Theorem 18.** *Let  $0 < k < n$ , let  $B$  be a closed convex subset of  $\mathbb{R}^n$  that contains no  $k$ -plane, and let  $\mathcal{P}$  be an open subset of  $\mathbb{L}_{n-k}^n$  such that  $B$  is not a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ . If  $C$  is a closed weak  $\mathcal{P}$ -imitation of  $B$  with  $C \neq B$ , then  $C \cap B$  contains a closed set that is homeomorphic to either*

- (i)  $\mathbb{R}^{k-1}$  or
- (ii)  $S^i \times \mathbb{R}^{k-i-1}$  for some  $i \in \{1, 2, \dots, k - 1\}$ .

*Proof.* Note that  $\mathcal{P} \neq \emptyset$  because  $B$  is not a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ . Since  $B$  is convex  $\dim B = \dim(\text{aff } B)$ , and we have by Lemma 13 that  $\dim B \geq k$ . We may now apply Theorem 17 to  $B$  and  $\mathcal{P}$  to obtain that there is a nonempty closed subset  $A$  of  $\mathcal{E}_k(B, \mathcal{P})$  that is homeomorphic to the boundary of a convex body in  $\mathbb{R}^k$ . By Remark 4 we have that  $A$  is homeomorphic to (i) or (ii), where the case  $S^0 \times \mathbb{R}^{k-1}$  is covered by option (i). It follows from Theorem 16 that  $A \subset \mathcal{E}_k(B, \mathcal{P}) \subset C \cap B$ .  $\square$

In particular, we have the following theorem which concerns projections onto hyperplanes. If  $u \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ , then we let  $\psi_u$  stand for the projection  $\psi_{\mathbb{R}u} : \mathbb{R}^n \rightarrow H_u$ .

**Theorem 19.** *Let  $B$  be a closed convex subset of  $\mathbb{R}^n$  that contains no hyperplane and let  $\mathcal{P}$  be an open subset of  $S^{n-1}$  such that  $\psi_u(B) \neq H_u$  for some  $u \in \mathcal{P}$ . If  $C$  is a closed weak  $\mathcal{P}$ -imitation of  $B$  with  $C \neq B$ , then  $C \cap B$  contains a closed set that is homeomorphic to either*

- (i)  $\mathbb{R}^{n-2}$  or
- (ii)  $S^i \times \mathbb{R}^{n-i-2}$  for some  $i \in \{1, 2, \dots, n-2\}$ .

*Proof.* We substitute  $k = n - 1$  in Theorem 18, and we note that we can replace the projective space  $\mathbb{L}_1^n$  by  $S^{n-1}$ .  $\square$

We obtain Theorem 1 as a corollary to Theorem 18.

*Proof of Theorem 1.* Let  $B = \overline{\langle C \rangle}$ . Since  $C$  is nonconvex we have that  $C \neq B$ . We have that  $C$  is a weak  $\mathcal{P}$ -imitation of  $B$  because

$$\overline{\psi_P(C)} \subset \overline{\psi_P(B)} = \overline{\psi_P(\langle C \rangle)} \subset \overline{\langle \psi_P(C) \rangle} \subset \overline{\langle \psi_P(C) \rangle} = \overline{\psi_P(C)}$$

for each  $P \in \mathcal{P}$ . If  $\psi_{P^*}(\langle C \rangle) \neq (P^*)^\perp$ , then  $\psi_{P^*}(B) \neq (P^*)^\perp$  because finite-dimensional affine spaces do not contain convex and dense proper subsets. Thus  $B$  is not a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ . Suppose that  $M$  is a  $k$ -plane in  $B$ . Select an  $x \in B^\circ$  and a  $y \in M$  and note that  $x - y + M$  is a  $k$ -plane in  $B^\circ$ . By finite-dimensionality we have  $B^\circ \subset \langle C \rangle$  which contradicts a premise of the theorem. Thus we have that  $B$  contains no  $k$ -plane. Now apply Theorem 18.  $\square$

*Remark 5.* In a forthcoming paper [2] we show that in Theorems 1, 18, and 19 the premise that  $\mathcal{P}$  is open can be relaxed to the condition that  $\mathcal{P} \subset \text{int } \overline{\mathcal{P}}$ .

The following corollary to Theorem 18 improves upon the main result in [5].

**Theorem 20.** *Let  $0 < k < n$ , let  $B$  be a compact convex subset of  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be a nonempty open subset of  $\mathbb{L}_{n-k}^n$ . If  $C$  is a closed weak  $\mathcal{P}$ -imitation of  $B$  with  $C \neq B$ , then  $C \cap B$  contains a  $(k-1)$ -sphere.*

*Proof.* Clearly, the compactum  $B$  contains no  $k$ -plane. Also every projection of  $B$  is compact, so  $B$  cannot be a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ . Apply Theorem 18 and note that  $S^{k-1}$  is the only compact space included in the options (i) and (ii).  $\square$

## 5. IMITATIONS

Theorem 16 states that every weak  $\mathcal{P}$ -imitation of a convex set  $B$  contains the set of extremal points  $\mathcal{E}_k(B, \mathcal{P})$ . In this section we show that  $B$  has ‘minimal imitations’, that is, sets that contain little else besides  $\mathcal{E}_k(B, \mathcal{P})$ . The next two results generalize Lemma 3 and Theorem 6 of [1]. Since the proof of [1, Lemma 3] does not generalize in a straightforward manner we have included a detailed argument for Lemma 21.

*Remark 6.* The following fact can be found in [8, §2.5]. If  $B$  is a closed convex set in  $\mathbb{R}^n$ , then there is a unique linear space  $\mathcal{L}_B \subset \mathbb{R}^n$  such that  $\text{cs } B = B \cap (\mathcal{L}_B)^\perp$  is line-free and  $B = \mathcal{L}_B + \text{cs } B$ . Note that  $\text{cs } B = \psi_{\mathcal{L}_B}(B)$ .

**Lemma 21.** *Let  $0 < k < n$ , let  $B$  be a closed convex subset of  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be a subset of  $\mathbb{L}_{n-k}^n$ . Then there exists a closed subset  $\mathcal{K}_k(B, \mathcal{P})$  of  $B$  such that  $\mathcal{K}_k(B, \mathcal{P}) \cap \partial B \subset \mathcal{E}_k(B, \mathcal{P})$  and  $\psi_P(\mathcal{K}_k(B, \mathcal{P})) = \psi_P(B)$  for every  $P \in \mathcal{P}$  that satisfies  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$ .*

*Proof.* Choose a coordinate system for  $\mathbb{R}^n$  such that  $\mathbf{0} \in B^\circ$ . We define the set  $S$  to be the closure of

$$\{tx : 0 \leq t \leq 1 \text{ and } x \in \mathcal{E}_k(B, \mathcal{P})\},$$

and we observe that  $S$  is a subset of  $B$ . Since  $\mathbf{0} \in B^\circ$  and  $\mathcal{E}_k(B, \mathcal{P})$  is closed we have  $S \cap \partial B \subset \mathcal{E}_k(B, \mathcal{P})$ . We now define the following subset of  $B^\circ$ :

$$T = \{\xi(x) : x \in B\},$$

where  $\xi(x) = \frac{\|x\|}{\|x\|+1}x$  for  $x \in \mathbb{R}^n$ . Let  $x_1, x_2, \dots$  be a sequence in  $B$  such that  $\lim_{n \rightarrow \infty} \xi(x_n) = y \in \mathbb{R}^n$ . If  $\lim_{i \rightarrow \infty} \|x_i\| = \infty$ , then  $\lim_{i \rightarrow \infty} \|\xi(x_i)\| = \infty \neq \|y\|$ . Thus  $x_1, x_2, \dots$  has a limit point  $x \in B$ . Note that  $\xi(x) = y$ , so we may conclude that  $T$  is closed. We define  $K = \mathcal{K}_k(B, \mathcal{P}) = S \cup T$ , and we note that  $K \subset B$ , that  $K \cap \partial B \subset \mathcal{E}_k(B, \mathcal{P})$ , and that  $K$  is a closed set such that  $tx \in K$  whenever  $x \in K$  and  $t \in [0, 1]$ .

Let  $P \in \mathcal{P}$  such that  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$  and let  $x \in B$  be arbitrary. It suffices to show that  $\psi_P(x) \in \psi_P(K)$ . We define an  $s \in [1, \infty]$  by

$$s = \sup\{t : t \geq 1 \text{ and } t\psi_P(x) \in \psi_P(B)\},$$

and we consider the following two cases:

Case I:  $s = \infty$  or  $s\psi_P(x) \notin \psi_P(B)$ . Then  $s > 1$ , and we may select sequences  $s_1, s_2, \dots$  in  $(1, s)$  and  $y_1, y_2, \dots$  in  $B$  such that  $\lim_{i \rightarrow \infty} s_i = s$  and  $y_i \in B$  with  $\psi_P(y_i) = s_i\psi_P(x)$  for each  $i \in \mathbb{N}$ . If the sequence  $y_1, y_2, \dots$  has a limit point  $y$ , then  $y \in B$  and  $\psi_P(y) = s\psi_P(x)$  if  $s < \infty$  or  $\psi_P(x) = \mathbf{0}$  if  $s = \infty$ . The first option is a violation of the premise of this case, and when  $\psi_P(x) = \mathbf{0} = \psi_P(\mathbf{0})$  we are done because  $\mathbf{0} \in K$ . Thus we may assume that  $\lim_{i \rightarrow \infty} \|y_i\| = \infty$ , and hence  $\lim_{i \rightarrow \infty} s_i \|y_i\| / (\|y_i\| + 1) = s$ . Choose an  $i \in \mathbb{N}$  with  $r = s_i \|y_i\| / (\|y_i\| + 1) > 1$ . Note that  $\xi(y_i) \in T \subset K$ , and hence  $\frac{1}{r}\xi(y_i) \in K$ . We have  $\psi_P(\frac{1}{r}\xi(y_i)) = \psi_P(\frac{1}{s_i}y_i) = \psi_P(x)$ , and Case I is proved.

Case II:  $s < \infty$  and  $s\psi_P(x) \in \psi_P(B)$ . In this case  $y = s\psi_P(x) \in \partial\psi_P(B)$ , so we can find a hyperplane  $H$  in  $P^\perp$  that is supporting to  $\psi_P(B)$  at  $y$ . Then  $F = (H + P) \cap B$  is a  $\mathcal{P}$ -face of  $B$  that meets  $y + P$ . Define the collection

$$\mathcal{D} = \{F' : F' \text{ is a derived face of } F \text{ such that } F' \cap (y + P) \neq \emptyset\}.$$

Since  $F \in \mathcal{D}$  we can select an element  $F'$  of  $\mathcal{D}$  with minimal dimension.

Suppose that  $\dim F' \geq k$  and note that

$$\begin{aligned} \dim((y + P) \cap \text{aff } F') &\geq \dim P + \dim F' - \dim(H + P) \\ &\geq (n - k) + k - (n - 1) = 1. \end{aligned}$$

Let  $\ell$  be a line through  $\mathbf{0}$  such that  $y' + \ell \subset (y + P) \cap \text{aff } F'$  for some  $y' \in (y + P) \cap F'$ . Since  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$  we have that  $\ell \cap \mathcal{L}_B = \{\mathbf{0}\}$ , and hence  $\ell' = \psi_{\mathcal{L}_B}(y' + \ell)$  is also a line. Since  $\text{cs } B$  is line-free we have that  $\ell'$  is not contained in  $\text{cs } B = \psi_{\mathcal{L}_B}(B)$ . Thus  $y' + \ell$  is neither contained in  $B$  nor in its subset  $F'$ . Since  $y' + \ell$  is a subset of  $\text{aff } F'$  that meets  $F'$ , the line contains a point  $z$  of  $\partial F'$ . Thus  $z$  is contained in some face  $G$  of  $F'$  and  $z \in y + P$ . We have  $G \in \mathcal{D}$  and  $\dim G < \dim F'$  in violation of the choice of  $F'$ . So we may conclude that  $\dim F' < k$ , and hence  $F' \subset \mathcal{E}_k(B, \mathcal{P})$ . Choose a point  $a \in F' \cap (y + P)$  and note that  $\frac{1}{s}a \in S \subset K$  and  $\psi_P(\frac{1}{s}a) = \psi_P(x)$ . The proof is complete.  $\square$



**Theorem 22** (Imitation Theorem). *Let  $0 < k < n$ , let  $B$  be a closed convex subset of  $\mathbb{R}^n$  with  $\dim B \neq k$ , and let  $\mathcal{P}$  be a subset of  $\mathbb{L}_{n-k}^n$ . Then there exists a closed  $\mathcal{P}$ -imitation  $C$  of  $B$  such that  $C \subset B$  and  $\dim(C \setminus \mathcal{E}_k(B, \mathcal{P})) \leq 0$ .*

*Proof.* If  $\mathcal{P} = \emptyset$ , then there is nothing to prove. Let  $P \in \mathcal{P}$ . If  $\dim B < k$ , then  $\dim(P + \text{aff } B) \leq n - 1$ , so  $P + \text{aff } B$  is contained in some hyperplane of  $\mathbb{R}^n$ . This means that  $B$  is a  $\mathcal{P}$ -face of itself and that  $\mathcal{E}_k(B, \mathcal{P}) = B$ , so we may choose  $C = B$ .

Assume now that  $\mathcal{P} \neq \emptyset$  and  $\dim B > k$ . As in [1, Theorem 6]  $C$  will have the form  $\mathcal{E}_k(B, \mathcal{P}) \cup Z_1 \cup Z_2$ , where  $Z_1$  and  $Z_2$  are countable unions of Cantor sets. Consider the locally compact space  $D = B \setminus \mathcal{E}_k(B, \mathcal{P})$  and its closed subset  $K = \mathcal{K}_k(B, \mathcal{P}) \setminus \mathcal{E}_k(B, \mathcal{P})$ ; see Lemma 21. Precisely as in the proof of [1, Theorem 6] we can construct a zero-dimensional subset  $Z_1$  of  $D$  such that  $\mathcal{E}_k(B, \mathcal{P}) \cup Z_1$  is closed in  $\mathbb{R}^n$  and every line in  $\text{aff } B$  that meets  $K$  also meets  $Z_1$ . The set  $Z_2$  is also constructed in the manner of [1], and it has the properties:  $Z_2$  is a zero-dimensional closed subset of  $B$  such that for every line  $\ell$  in  $\mathcal{L}_B$  and point  $x \in B$  we have  $(x + \ell) \cap Z_2 \neq \emptyset$ .

Let  $P \in \mathcal{P}$  and  $x \in B$  be arbitrary. It now suffices to show that  $\psi_P(x) \in \psi_P(C)$ . Assume first that  $P \cap \mathcal{L}_B \neq \{\mathbf{0}\}$  and hence that  $P \cap \mathcal{L}_B$  contains a line  $\ell$  through  $\mathbf{0}$ . Since  $x + \ell$  intersects  $Z_2$  we have  $\psi_P(x) \in \psi_P(Z_2) \subset \psi_P(C)$ . Now let  $P \cap \mathcal{L}_B = \{\mathbf{0}\}$  and note that we may apply Lemma 21 to find that  $\psi_P(B) = \psi_P(\mathcal{K}_k(B, \mathcal{P}))$ ; thus  $(x + P) \cap \mathcal{K}_k(B, \mathcal{P}) \neq \emptyset$ . If  $(x + P) \cap \mathcal{E}_k(B, \mathcal{P}) \neq \emptyset$ , then we are done. So we may assume that  $(x + P) \cap K \neq \emptyset$ . Since  $\dim B > k$  we have  $\dim((x + P) \cap \text{aff } B) \geq 1$ , and we can find a line  $\ell$  in  $(x + P) \cap \text{aff } B$  that meets  $K$ . Thus  $\ell$  must also meet  $Z_1$  in a point  $z$ , and we have  $\psi_P(x) = \psi_P(z) \in \psi_P(Z_1) \subset \psi_P(C)$ .  $\square$

We now show that the  $\mathcal{P}$ -extremal points of  $B$  are precisely the points that are included in every closed  $\mathcal{P}$ -imitation of  $B$  and hence that Theorem 16 is sharp.

**Theorem 23.** *Let  $0 < k < n$ , let  $B$  be a closed convex set in  $\mathbb{R}^n$  with  $\dim B \neq k$ , and let  $\mathcal{P}$  be a nonempty open subset of  $\mathbb{L}_{n-k}^n$ . Then*

$$\begin{aligned} \mathcal{E}_k(B, \mathcal{P}) &= \bigcap \{C : C \text{ is a closed weak } \mathcal{P}\text{-imitation of } B\} \\ &= \bigcap \{C : C \text{ is a closed } \mathcal{P}\text{-imitation of } B\}. \end{aligned}$$

*Proof.* Theorem 16 shows that

$$\begin{aligned} \mathcal{E}_k(B, \mathcal{P}) &\subset \bigcap \{C : C \text{ is a closed weak } \mathcal{P}\text{-imitation of } B\} \\ &\subset \bigcap \{C : C \text{ is a closed } \mathcal{P}\text{-imitation of } B\}. \end{aligned}$$

Theorem 22 guarantees that there exist closed  $\mathcal{P}$ -imitations  $C$  of  $B$  with  $C = \mathcal{E}_k(B, \mathcal{P}) \cup Z_1 \cup Z_2$ , where  $Z_1$  and  $Z_2$  are constructed as countable unions of Cantor sets. There is a lot of freedom in choosing the Cantor sets, specifically it is easily seen that given any point  $x \in \mathbb{R}^n$  it can be arranged that  $Z_1 \cup Z_2$  avoids that point. This observation proves the theorem.  $\square$

*Remark 7.* We now explain why the case  $\dim B = k$  is excluded in Theorems 22 and 23. Let  $B$  be convex and closed in  $\mathbb{R}^n$  with  $\dim B = k$  and let  $\mathcal{P}$  be a nonempty open subset of  $\mathbb{L}_{n-k}^n$ . It is easily seen that now  $\mathcal{E}_k(B, \mathcal{P}) = \partial B$ . Select a coordinate system such that  $\mathbf{0} \in B^\circ$ . Let  $C$  be a closed weak  $\mathcal{P}$ -imitation of  $B$  such that  $C \subset \text{aff } B$ . Since  $\mathcal{P}$  is open and nonempty and  $\text{aff } B = k$ , we can find a  $P \in \mathcal{P}$  such that  $P \cap \text{aff } B = \{\mathbf{0}\}$ . Then  $\psi_P \upharpoonright \text{aff } B : \text{aff } B \rightarrow P^\perp$  is an isomorphism, and hence

$\overline{\psi_P(C)} = \overline{\psi_P(B)}$  implies  $C = B$ . We have  $\dim(C \setminus \mathcal{E}_k(B, \mathcal{P})) = \dim B^\circ = k$ , and thus Theorem 22 is false whenever  $\dim B = k$ .

Consider now Theorem 23 and Theorem 22', that is, Theorem 22 without the requirement that  $C \subset B$ . Then we have two cases.

Case I: there is an  $x \notin \text{aff } B$  such that  $\psi_P(x) \in \psi_P(B)$  for every  $P \in \mathcal{P}$ . In this case the conclusions of Theorems 22' and 23 are valid for  $B$ . Put  $B' = \overline{B \cup \{x\}}$  and note that  $B'$  is a closed and convex weak  $\mathcal{P}$ -imitation of  $B$  with  $\dim B' = k+1$ . By Lemma 9 we have  $\mathcal{E}_k(B, \mathcal{P}) = \mathcal{E}_k(B', \mathcal{P})$ . We can apply both Theorems 22 and 23 to  $B'$  and reach the desired conclusion for  $B$ .

Case II: otherwise. In this case the conclusions of Theorems 22' and 23 are always false. Let  $C$  be a closed weak  $\mathcal{P}$ -imitation of  $B$  and assume that there is an  $x \in C \setminus \text{aff } B$ . Then there is a  $P_1 \in \mathcal{P}$  such that  $\psi_{P_1}(\frac{1}{2}x) \notin \psi_{P_1}(B)$ . Since  $\mathbf{0} \in B^\circ$  it is easily verified that  $\psi_{P_1}(x) \notin \overline{\psi_{P_1}(B)}$ , and hence  $C$  is not a weak  $\mathcal{P}$ -imitation of  $B$ . Thus we may conclude that  $C \subset \text{aff } B$ , and by the argument above we have  $C = B$ . In this case, just as in the case  $\dim B < k$ , the set  $B$  has only itself as a closed weak  $\mathcal{P}$ -imitation and Theorem 18 is essentially void. Moreover,  $C \setminus \mathcal{E}_k(B, \mathcal{P}) = B^\circ$  and the conclusions of Theorems 22' and 23 are invalid.

We show that the two assumptions in Theorem 18 that  $B$  contains no  $k$ -plane and that  $B$  is not a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$  are necessary conditions (except in the case  $k = 1$  when the theorem is void).

**Proposition 24.** *Let  $0 < k < n$ , let  $B$  be a closed convex set in  $\mathbb{R}^n$ , and let  $\mathcal{P}$  be a nonempty open subset of  $\mathbb{L}_{n-k}^n$ .*

- (a) *If  $B$  is a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ , then  $B$  has a closed  $\mathcal{P}$ -imitation that is zero-dimensional.*
- (b) *If  $B$  contains a  $k$ -plane, then either  $B$  has a closed  $\mathcal{P}$ -imitation that is zero-dimensional or  $B$  is a  $k$ -plane that admits no closed weak  $\mathcal{P}$ -imitation other than itself.*

*Proof.* Assume first that  $B$  is a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$ . Let  $P \in \mathcal{P}$  and note that  $\dim B \geq \dim \psi_P(B) = \dim P^\perp = k$ . Then by Theorem 17 we have  $\mathcal{E}_k(B, \mathcal{P}) = \emptyset$ . If  $\dim B > k$ , then  $B$  has a zero-dimensional and closed  $\mathcal{P}$ -imitation by Theorem 22. If  $\dim B = k$ , then since  $B$  is a  $\mathcal{P}$ -imitation of  $\mathbb{R}^n$  we are in Case I of Remark 7 and  $B$  also has a zero-dimensional and closed  $\mathcal{P}$ -imitation.

Assume that  $B$  contains a  $k$ -plane. Then by Theorem 17 we have  $\mathcal{E}_k(B, \mathcal{P}) = \emptyset$ . If  $\dim B > k$ , then  $B$  has a zero-dimensional and closed  $\mathcal{P}$ -imitation by Theorem 22. So we may assume that  $B$  is a  $k$ -plane. If  $B' = \bigcap \{B + P : P \in \mathcal{P}\}$  is not equal to  $B$ , then we are in Case I of Remark 7 and  $B$  admits a zero-dimensional and closed  $\mathcal{P}$ -imitation. So assume that  $B' = B$  and let  $C$  be a closed weak  $\mathcal{P}$ -imitation of  $B$ . Then clearly,  $C \subset B'$ . Since  $B$  is a  $k$ -plane and  $\mathcal{P}$  is open, there is a  $P \in \mathcal{P}$  with  $B + P = \mathbb{R}^n$ . Then  $\psi_P|_B: B \rightarrow P^\perp$  is an isomorphism. Since  $\overline{\psi_P(B)} = \overline{\psi_P(C)}$  and  $C \subset B$ , we have that  $C = B$ .  $\square$

*Remark 8.* Example 4 in [1] shows that in Theorem 1 we may not replace the condition  $\psi_{P^*}(\langle C \rangle) \neq (P^*)^\perp$  by the weaker statement  $\psi_{P^*}(C) \neq (P^*)^\perp$ .

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